

**MAT 402 S26**  
**PROBLEM SET 3**

**I.** Let  $A$  be an affine space. Recall that

- (i) a subset  $\Omega \subset A$  is *convex* if for every  $x, y \in \Omega$  and every  $\lambda \in (0, 1)$  the point  $x + \lambda(y - x)$  lies in  $\Omega$ , and
- (ii) a function  $f : \Omega \rightarrow \mathbb{R}$  is *convex* if  $\Omega$  and the supergraph  $\Gamma^+(f) := \{(x, t) \in \Omega \times \mathbb{R} ; f(x) \leq t\}$  are convex sets.

Show that the following are equivalent.

- a.  $f : \Omega \rightarrow \mathbb{R}$  is convex.
- b.  $f(x + \lambda(y - x)) \leq f(x) + \lambda(f(y) - f(x))$  for every  $x, y \in \Omega$  and every  $\lambda \in (0, 1)$ .
- c. For every  $x \in A$  and every  $v \in T(A)$  the function

$$\mathbb{R} - \{0\} \ni t \mapsto \frac{f(x + tv) - f(x)}{t}$$

is an increasing function.

**II.** In class we stated the following theorem.

**THEOREM (Hyperplane separation theorem).** *If  $\Omega_1$  and  $\Omega_2$  are disjoint convex sets in an affine space  $A$  then there is a continuous affine function  $f : A \rightarrow \mathbb{R}$  such that*

$$f|_{\Omega_+} \geq 0 \quad \text{and} \quad f|_{\Omega_-} \leq 0.$$

*If moreover  $\Omega_+$  is closed and  $\Omega_-$  is compact then  $f$  can be chosen so that*

$$f|_{\Omega_+} \geq 1 \quad \text{and} \quad f|_{\Omega_-} \leq -1.$$

Show that in every affine space  $A$  there are disjoint convex sets  $\Omega_+$  and  $\Omega_-$  such that if  $f$  is any affine function satisfying  $f|_{\Omega_+} \geq 0$  and  $f|_{\Omega_-} \leq 0$  then equality holds for some point  $x \in \Omega_+$  and some point  $y \in \Omega_-$ .

**III.** For a subset  $\Omega$  of an affine space, define its *indicator function*

$$\mathbf{I}_\Omega(x) := \begin{cases} 0, & x \in \Omega \\ +\infty, & x \in A - \Omega \end{cases}.$$

Show that  $\Omega$  is a convex set if and only if its indicator function is a convex function.

**IV.** Let  $\Omega$  be a convex subset of an affine set. Show that for any function  $f : \Omega \rightarrow \mathbb{R}$  the *affine envelope*  $\mathcal{E}(f) : \Omega \rightarrow \mathbb{R}$  defined by

$$\mathcal{E}(f)(x) := \sup \{ \phi(x) ; \phi : A \rightarrow \mathbb{R} \text{ affine and } \phi \leq f \}$$

is convex, and that for any convex function  $g \leq f$  we have  $g \leq \mathcal{E}(f)$ .

**V.** For  $\alpha \in (1, \infty)$  let  $f_\alpha(x) = \frac{|x|^\alpha}{\alpha}$ ,  $x \in \mathbb{R}$ . Find the Legendre transform  $f_\alpha^*$  of  $f_\alpha$ .